4.9

Graphing of Rational Functions (Honors Topic)

Rational Functions:

Let *f* and g be polynomial functions with g(x) $\neq$ 0. The function given by $r(x)=\frac{f(x)}{g(x)}$ is a rational function.

Finding the Domain of a Rational Function:

Example: Find the domain of *f* and use limits to describe its behavior at value(s) of *x* not in its domain.

$$f(x)=\frac{1}{x+2}$$

The domain of *f* is all real numbers $x \neq -2$. This tells us that there is a vertical asymptote at x = -2.

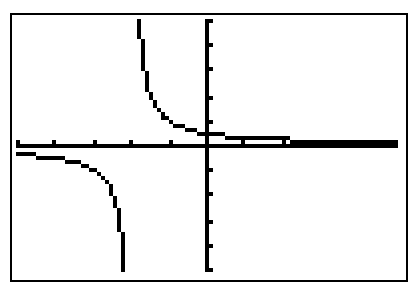
Viewing the graph below we can determine the behavior of the graph as it approaches the asymptote from the right side of the asymptote and the left side of the asymptote. We do this by using limits.

Asymptotes – describe behavior of graph at its horizontal or vertical extremities:

$\lim_{x\to a^+} f(x)$ means "x approaches a from the right"

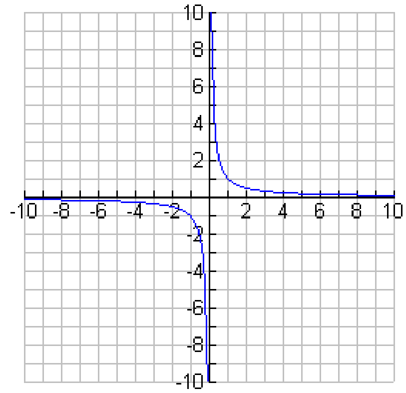
$\lim_{x\to a^-} f(x)$ means "x approaches a from the left"

Here we have $\lim_{x\to -2^+} f(x)=\infty$ and $\lim_{x\to -2^-} f(x)=-\infty$.



<u>The Reciprocal Function</u>:

f(x) = $\frac{1}{x}$



Domain: $\left(-\infty,0\right)\cup\left(0,\infty\right)$

Range: $\left(-\infty,0\right)\cup\left(0,\infty\right)$

Continuity: All x $\neq$ 0

Decreasing on $\left(-\infty,0\right)\, and \,\left(0,\infty\right)$

Symmetric with respect to the origin (odd)

Unbounded

No local extrema

Horizontal asymptote: y = 0, Vertical asymptote: x = 0

End Behavior: $\lim_{x\to-\infty} f(x)=0$ and $\lim_{x\to\infty} f(x)=0$

<u>Transformations of the Reciprocal Function</u>:

Remember the reciprocal function is y = $\frac{1}{x}$.

The graph of any rational function of the form $g(x)=\frac{ax+b}{cx+d}$ can be obtained through transformations of the reciprocal function. If the degree of the numerator is greater than or equal to the degree of the denominator, use polynomial division to rewrite the rational function.

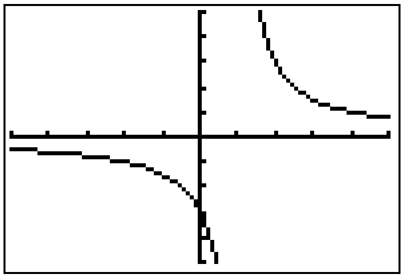
Example:

Describe how the graph of the given function can be obtained from the graph of the reciprocal function $f(x)=\frac{1}{x}$. Identify the horizontal and vertical asymptotes and use limits to describe the corresponding behavior. Sketch the graph of the function. Transformed reciprocal function: $g(x)=a\cdot f\left(\frac{1}{x-h}\right)+k$

a) $g(x)=\frac{3}{x-1}$

This can be rewritten as $g(x)=\dfrac{3}{x-1}=3\left(\dfrac{1}{x-1}\right)$. The transformation is a vertical stretch by a factor of 3 and a horizontal shift right 1. Vertical asymptote at *x* = 1, horizontal asymptote at *y* = 0. $\lim_{x\to 1^-} g(x)=-\infty$ and $\lim_{x\to 1^+} g(x)=\infty$.

Graph:



Try #1 on page 94.

b) $h(x)=\dfrac{4x-1}{x+2}$

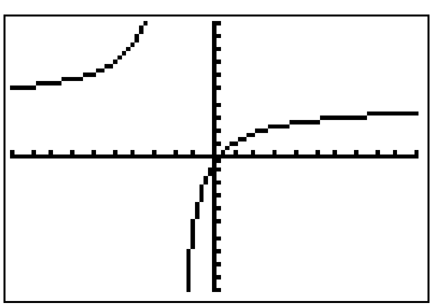
To find the transformation here we need to use long division.

$$\begin{array}{r} 4 \\ x+2\overline{)4x-1} \\ \underline{-4x-8} \\ -9 \end{array}$$

So, $h(x)=\dfrac{4x-1}{x+2}$ can be written as $h(x)=\dfrac{4x-1}{x+2}=4-\dfrac{9}{x+2}=-9\left(\dfrac{1}{x+2}\right)+4$

This is a horizontal shift left 2, vertical stretch by a factor of 9, a reflection across the *x*-axis, and then a vertical shift up 4. Vertical asymptote at *x* = -2, horizontal asymptote at *y* = 4. $\lim_{x\to -2^-} h(x)=\infty$ and $\lim_{x\to -2^+} h(x)=-\infty$.

Graph:



Try #6 on pg. 94

Limits and Asymptotes

Graphical Features of a Rational Function

$$y = \frac{f(x)}{d(x)} = \frac{(a_n x^n + ...)}{(b_m x^m + ...)}$$

1. End behavior asymptote:

If $n < m$, the end behavior asymptote is the horizontal asymptote y = 0.

If $n = m$, the end behavior asymptote is the horizontal asymptote $y = \frac{a_n}{b_m}$.

If n > m, the end behavior asymptote is the quotient polynomial function $y = q(x)$, where $f(x) = d(x)q(x) + r(x)$. There is no horizontal asymptote. (**Slant asymptote**)

2. Vertical asymptotes: These occur at the zeros of the denominator provided that the zeros are not also zeros of the numerator or equal or greater multiplicity.

3. x-intercepts: These occur at the zeros of the numerator, which are not also zeros of the denominator .

4. y-intercept: This is the value of f(0), if defined.

Example:

Find the asymptotes and intercepts of the function $f(x) = \frac{x^3}{x^2 - 4}$.

Need to find **slant asymptote** and **vertical asymptotes**. Slant asymptote we find by using long division:

$f(x) = \frac{x^3}{x^2 - 4} = x + \frac{4x}{x^2 - 4}$; Asymptote at $y = x$. Vertical asymptotes come from factoring the denominator.

Vertical asymptotes at $x = 2$, and $x = -2$. The y-intercept comes from making the numerator zero, which gives (0, 0) as the both the *y*-intercept and *x*-intercept here.

See #18 pg. 94